MATHCHAPTER C

Vectors

A vector is a quantity that has both magnitude and direction. Examples of vectors are position, force, velocity, and momentum. We specify the position of something, for example, by giving not only its distance from a certain point but also its direction from that point. We often represent a vector by an arrow, where the length of the arrow is the magnitude of the vector and its direction is the same as the direction of the vector.

Two vectors can be added together to get a new vector. Consider the two vectors $\mathbf{u}$ and $\mathbf{v}$ in Figure C.1. (We denote vectors by boldface symbols.) To find $\mathbf{w} = \mathbf{u} + \mathbf{v}$, we place the tail of $\mathbf{u}$ at the tip of $\mathbf{v}$ and then draw $\mathbf{w}$ from the tail of $\mathbf{v}$ to the tip of $\mathbf{u}$, as shown in the figure. We could also have placed the tail of $\mathbf{u}$ at the origin and then placed the tail of $\mathbf{v}$ at the tip of $\mathbf{u}$ and drawn $\mathbf{w}$ from the tail of $\mathbf{u}$ to the tip of $\mathbf{v}$. As Figure C.1 indicates, we get the same result either way, so we see that

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \tag{C.1}$$

Vector addition is commutative.

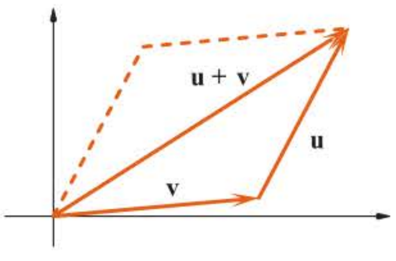

FIGURE C.1
An illustration of the addition of two vectors, $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} = \mathbf{w}$.

To subtract two vectors, we draw one of them in the opposite direction and then add it to the other. Writing a vector in its opposite direction is equivalent to forming the vector $-\mathbf{v}$. Thus, mathematically we have

$$\mathbf{t} = \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) \tag{C.2}$$

Generally, a number a times a vector is a new vector that is parallel to **u** but whose length is a times the length of **u**. If a is positive, then a**u** lies in the same direction as **u**, but if a is negative, then a**u** lies in the opposite direction.

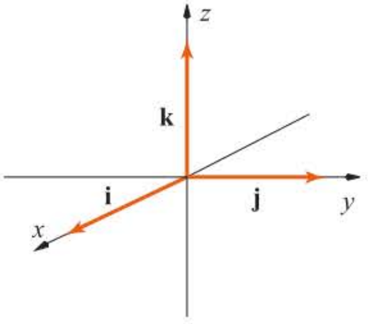

FIGURE C.2
The fundamental unit vectors **i**, **j**, and **k** of a cartesian coordinate system.

A useful set of vectors are the vectors that are of unit length and point along the positive x, y, and z axes of a cartesian coordinate system. These *unit vectors* (unit length), which we designate by **i**, **j**, and **k**, respectively, are shown in Figure C.2. We shall always draw a cartesian coordinate system so that it is right-handed. A *right-handed coordinate system* is such that when you curl the four fingers of your right hand from **i** to **j**, your thumb points along **k** (Figure C.3).

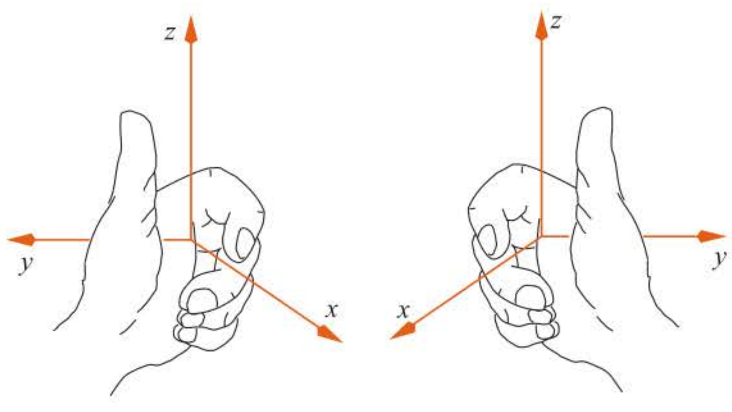

FIGURE C.3
An illustration of a right-handed cartesian coordinate system (right) and a left-handed cartesian coordinate system (left).

Any three-dimensional vector **u** can be described in terms of these unit vectors,

$$\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k} \tag{C.3}$$

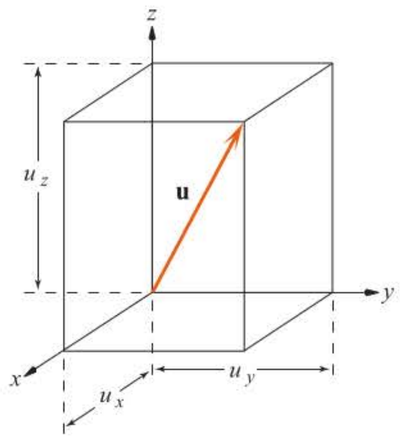

FIGURE C.4
The components of a vector **u** are its projections along the x, y, and z axes, showing that the length of **u** is equal to $(u_x^2 + u_y^2 + u_z^2)^{1/2}$.

where, for example, $u_x\,\mathbf{i}$ is u_x units long and lies in the direction of **i**. The quantities u_x, u_y, and u_z in Equation C.3 are the *components* of **u**. They are the projections of **u** along the respective cartesian axes (Figure C.4). In terms of components, the sum or difference of two vectors is given by

$$\mathbf{u} \pm \mathbf{v} = (u_x \pm v_x)\,\mathbf{i} + (u_y \pm v_y)\,\mathbf{j} + (u_z \pm v_z)\,\mathbf{k} \tag{C.4}$$

Figure C.4 shows that the length of **u** is given by

$$u = |\mathbf{u}| = (u_x^2 + u_y^2 + u_z^2)^{1/2} \tag{C.5}$$

EXAMPLE C–1
If $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, then what is the length of $\mathbf{u} + \mathbf{v}$?

SOLUTION: Using Equation C.4, we have

$$\mathbf{u} + \mathbf{v} = (2-1)\mathbf{i} + (-1+2)\mathbf{j} + (3-1)\mathbf{k} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

and using Equation C.5 gives

$$|\mathbf{u} + \mathbf{v}| = (1^2 + 1^2 + 2^2)^{1/2} = \sqrt{6}$$

There are two ways to form the product of two vectors, and both have many applications in physical chemistry. One way yields a scalar quantity (in other words, just a number), and the other yields a vector. Not surprisingly, we call the result of the first method a *scalar product* and the result of the second method a *vector product*.

The scalar product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta \tag{C.6}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$. Note from the definition that

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \tag{C.7}$$

Taking a scalar product is a *commutative operation*. The dot between $\mathbf{u}$ and $\mathbf{v}$ is such a standard notation that $\mathbf{u} \cdot \mathbf{v}$ is often called the *dot product* of $\mathbf{u}$ and $\mathbf{v}$. The dot products of the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are

$$\begin{gathered} \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = |1||1| \cos 0^\circ = 1 \\ \mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{i} = \mathbf{i} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{j} = |1||1| \cos 90^\circ = 0 \end{gathered} \tag{C.8}$$

We can use Equations C.8 to evaluate the dot product of two vectors:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k}) \cdot (v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k}) \\ &= u_x v_x\,\mathbf{i} \cdot \mathbf{i} + u_x v_y\,\mathbf{i} \cdot \mathbf{j} + u_x v_z\,\mathbf{i} \cdot \mathbf{k} \\ &\quad + u_y v_x\,\mathbf{j} \cdot \mathbf{i} + u_y v_y\,\mathbf{j} \cdot \mathbf{j} + u_y v_z\,\mathbf{j} \cdot \mathbf{k} \\ &\quad + u_z v_x\,\mathbf{k} \cdot \mathbf{i} + u_z v_y\,\mathbf{k} \cdot \mathbf{j} + u_z v_z\,\mathbf{k} \cdot \mathbf{k} \end{aligned}$$

and so

$$\mathbf{u} \cdot \mathbf{v} = u_x v_x + u_y v_y + u_z v_z \tag{C.9}$$

EXAMPLE C–2
Find the length of $\mathbf{u} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$.

SOLUTION: Equation C.9 with $\mathbf{u} = \mathbf{v}$ gives

$$\mathbf{u} \cdot \mathbf{u} = u_x^2 + u_y^2 + u_z^2 = |\mathbf{u}|^2$$

Therefore,

$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2} = (4 + 1 + 9)^{1/2} = \sqrt{14}$$

EXAMPLE C–3

Find the angle between the two vectors $\mathbf{u} = \mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$ and $\mathbf{v} = \mathbf{j} - \mathbf{k}$.

SOLUTION: We use Equation C.6, but first we must find

$$|\mathbf{u}| = (\mathbf{u}\cdot\mathbf{u})^{1/2} = (1+9+1)^{1/2} = \sqrt{11}$$

$$|\mathbf{v}| = (\mathbf{v}\cdot\mathbf{v})^{1/2} = (0+1+1)^{1/2} = \sqrt{2}$$

and

$$\mathbf{u}\cdot\mathbf{v} = 0+3+1 = 4$$

Therefore,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{4}{\sqrt{22}} = 0.8528$$

or $\theta = 31.48^\circ$.

Because $\cos 90^\circ = 0$, the dot product of vectors that are perpendicular to each other is equal to zero. For example, the dot products between the **i**, **j**, and **k** cartesian unit vectors are equal to zero, as Equation C.8 says.

EXAMPLE C–4

Show that the vectors $\mathbf{v}_1 = \frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$, $\mathbf{v}_2 = \frac{1}{\sqrt{6}}\,\mathbf{i} - \frac{2}{\sqrt{6}}\,\mathbf{j} + \frac{1}{\sqrt{6}}\,\mathbf{k}$, and $\mathbf{v}_3 = -\frac{1}{\sqrt{2}}\,\mathbf{i} + \frac{1}{\sqrt{2}}\,\mathbf{k}$ are of unit length and are mutually perpendicular.

SOLUTION: The lengths are given by

$$(\mathbf{v}_1\cdot\mathbf{v}_1)^{1/2} = \left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)^{1/2} = 1$$

$$(\mathbf{v}_2\cdot\mathbf{v}_2)^{1/2} = \left(\frac{1}{6}+\frac{4}{6}+\frac{1}{6}\right)^{1/2} = 1$$

$$(\mathbf{v}_3\cdot\mathbf{v}_3)^{1/2} = \left(\frac{1}{2}+0+\frac{1}{2}\right)^{1/2} = 1$$

The dot products between the different vectors are

$$\mathbf{v}_1\cdot\mathbf{v}_2 = \frac{1}{\sqrt{18}} - \frac{2}{\sqrt{18}} + \frac{1}{\sqrt{18}} = 0$$

$$\mathbf{v}_1 \cdot \mathbf{v}_3 = -\frac{1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}} = 0$$

$$\mathbf{v}_2 \cdot \mathbf{v}_3 = -\frac{1}{\sqrt{12}} + 0 + \frac{1}{\sqrt{12}} = 0$$

None of the vector operations that we have used so far are limited to two or three dimensions. We can easily generalize Equation C.9 to N dimensions by writing

$$\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{N} u_j v_j \tag{C.10}$$

The length of an N-dimensional vector is given by

$$l = (\mathbf{u} \cdot \mathbf{u})^{1/2} = \left(\sum_{j=1}^{N} u_j^2 \right)^{1/2} \tag{C.11}$$

If the dot product of two N-dimensional vectors is equal to zero, then we say that the two vectors are *orthogonal*. Thus, the term orthogonal is just a generalization of perpendicular. Furthermore, if the length of a vector is equal to 1, then the vector is said to be *normalized*. A set of mutually orthogonal vectors that are also normalized is said to be *orthonormal*. It is common notation to represent N-dimensional vectors by just listing their components within parentheses. Problem C–7 has you show that the set of vectors $(1/\sqrt{3},\ 1/\sqrt{3},\ 0,\ 1/\sqrt{3})$, $(1/\sqrt{3},\ -1/\sqrt{3},\ 1/\sqrt{3},\ 0)$, $(0,\ 1/\sqrt{3},\ 1/\sqrt{3},\ -1/\sqrt{3})$, and $(1/\sqrt{3},\ 0,\ -1/\sqrt{3},\ -1/\sqrt{3})$ is orthonormal.

One application of a dot product involves the definition of work. Recall that work is defined as force times distance, where "force" means the component of force that lies in the same direction as the displacement. If we let $\mathbf{F}$ be the force and $\mathbf{d}$ be the displacement, then work is defined as

$$\text{work} = \mathbf{F} \cdot \mathbf{d} \tag{C.12}$$

We can write Equation C.12 as $(F \cos\theta)(d)$ to emphasize that $F \cos\theta$ is the component of $\mathbf{F}$ in the direction of $\mathbf{d}$ (Figure C.5).

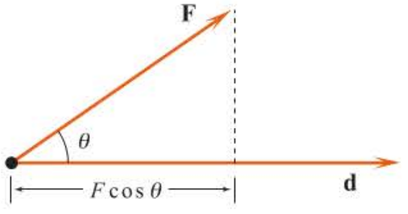

FIGURE C.5
Work is defined as $w = \mathbf{F} \cdot \mathbf{d}$, or $(F \cos\theta)d$, where $F \cos\theta$ is the component of $\mathbf{F}$ along $\mathbf{d}$.

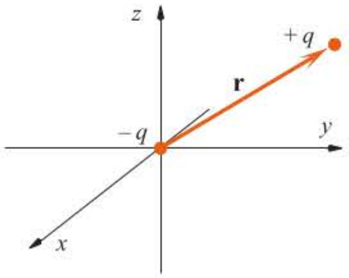

FIGURE C.6
A dipole moment is a vector that points from a negative charge, $-q$, to a positive charge, $+q$, and whose magnitude is qr.

Another important application of a dot product involves the interaction of a dipole moment with an electric field. You may have learned in organic chemistry that the separation of opposite charges in a molecule gives rise to a dipole moment, which is often indicated by an arrow pointing from the negative charge to the positive charge. For example, because a chlorine atom is more electronegative than a hydrogen atom, HCl has a dipole moment, which we indicate by writing $\overleftarrow{\text{HCl}}$. Strictly speaking, a dipole moment is a vector quantity whose magnitude is equal to the product of the positive charge and the distance between the positive and negative charges and whose direction is from the negative charge to the positive charge. Thus, for the two separated charges illustrated in Figure C.6, the dipole moment $\boldsymbol{\mu}$ is equal to

$$\boldsymbol{\mu} = q\,\mathbf{r}$$

We will learn later that if we apply an electric field $\mathbf{E}$ to a dipole moment, then the potential energy of interaction will be

$$V = -\boldsymbol{\mu} \cdot \mathbf{E} \tag{C.13}$$

The vector product of two vectors is a vector defined by

$$\mathbf{u} \times \mathbf{v} = |\mathbf{u}||\mathbf{v}|\,\mathbf{c}\sin\theta \tag{C.14}$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ and $\mathbf{c}$ is a unit vector perpendicular to the plane formed by $\mathbf{u}$ and $\mathbf{v}$. The direction of $\mathbf{c}$ is given by the right-hand rule: If the four fingers of your right hand curl from $\mathbf{u}$ to $\mathbf{v}$, then $\mathbf{c}$ lies along the direction of your thumb. (See Figure C.3 for a similar construction.) The notation given in Equation C.14 is so commonly used that the vector product is usually called the *cross product*. Because the direction of $\mathbf{c}$ is given by the right-hand rule, the cross product operation is not commutative, and, in particular

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} \tag{C.15}$$

The cross products of the cartesian unit vectors are

$$\begin{aligned}\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}&=|1||1|\,\mathbf{c}\sin 0^\circ=0\\ \mathbf{i}\times\mathbf{j}=-\mathbf{j}\times\mathbf{i}&=|1||1|\,\mathbf{k}\sin 90^\circ=\mathbf{k}\\ \mathbf{j}\times\mathbf{k}&=-\mathbf{k}\times\mathbf{j}=\mathbf{i}\\ \mathbf{k}\times\mathbf{i}&=-\mathbf{i}\times\mathbf{k}=\mathbf{j}\end{aligned}\tag{C.16}$$

In terms of components of **u** and **v**, we have (Problem C–10)

$$\mathbf{u}\times\mathbf{v}=(u_yv_z-u_zv_y)\,\mathbf{i}+(u_zv_x-u_xv_z)\,\mathbf{j}+(u_xv_y-u_yv_x)\,\mathbf{k}\tag{C.17}$$

Equation C.17 can be conveniently expressed as a determinant (see MathChapter E):

$$\mathbf{u}\times\mathbf{v}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ u_x&u_y&u_z\\ v_x&v_y&v_z\end{vmatrix}\tag{C.18}$$

Equations C.17 and C.18 are equivalent.

EXAMPLE C–5
Given $\mathbf{u}=-2\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{v}=3\mathbf{i}-\mathbf{j}+\mathbf{k}$, determine $\mathbf{w}=\mathbf{u}\times\mathbf{v}$.

SOLUTION: Using Equation C.17, we have

$$\begin{aligned}\mathbf{w}&=[(1)(1)-(1)(-1)]\,\mathbf{i}+[(1)(3)-(-2)(1)]\,\mathbf{j}+[(-2)(-1)-(1)(3)]\,\mathbf{k}\\ &=2\,\mathbf{i}+5\,\mathbf{j}-\mathbf{k}\end{aligned}$$

One physically important application of a cross product involves the definition of angular momentum. If a particle has a momentum $\mathbf{p}=m\mathbf{v}$ at a position **r** from a fixed point (as in Figure C.7), then its *angular momentum* is defined by

$$\mathbf{l}=\mathbf{r}\times\mathbf{p}\tag{C.19}$$

Note that the angular momentum is a vector perpendicular to the plane formed by **r** and **p** (Figure C.8). In terms of components, **l** is equal to (see Equation C.17)

$$\mathbf{l}=(yp_x-xp_y)\,\mathbf{i}+(zp_x-xp_z)\,\mathbf{j}+(xp_y-yp_x)\,\mathbf{k}\tag{C.20}$$

We will see that angular momentum plays an important role in quantum mechanics.

Another example that involves a cross product is the equation that gives the force **F** on a particle of charge q moving with velocity **v** through a magnetic field **B**:

$$\mathbf{F}=q(\mathbf{v}\times\mathbf{B})$$

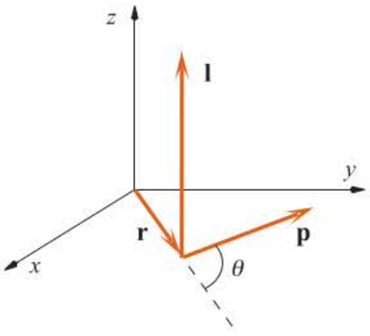

FIGURE C.7
The angular momentum of a particle of momentum **p** and position **r** from a fixed center is a vector perpendicular to the plane formed by **r** and **p** and in the direction of **r** × **p**.

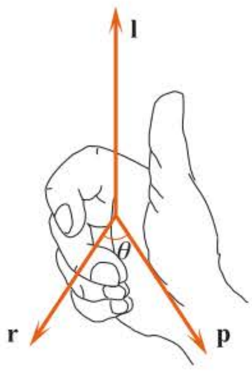

FIGURE C.8
Angular momentum is a vector quantity that lies perpendicular to the plane formed by **r** and **p** and is directed such that the vectors **r**, **p**, and **l** form a right-handed coordinate system.

Note that the force is perpendicular to **v**, and so the effect of **B** is to cause the motion of the particle to curve, not to speed up or slow down.

We can also take derivatives of vectors. Suppose that the components of momentum, **p**, depend upon time. Then

$$\frac{d\mathbf{p}(t)}{dt} = \frac{dp_x(t)}{dt}\,\mathbf{i} + \frac{dp_y(t)}{dt}\,\mathbf{j} + \frac{dp_z(t)}{dt}\,\mathbf{k} \tag{C.21}$$

(There are no derivatives of **i**, **j**, and **k** because they are fixed in space.) Newton's law of motion is

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{C.22}$$

This law is actually three separate equations, one for each component. Because $\mathbf{p} = m\mathbf{v}$, if m is a constant, we can write Newton's equation as

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}$$

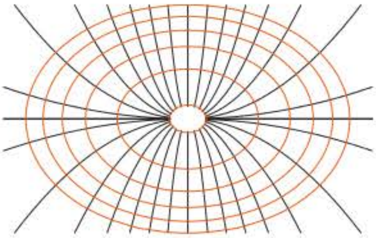

FIGURE C.9
A set of level curves (orange) for the surface $z = f(x, y)$ and the path $\nabla f(x, y)$ (black), which follows the direction of steepest descent.

Furthermore, because $\mathbf{v} = d\mathbf{r}/dt$, we can also express Newton's equations as

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} \tag{C.23}$$

Once again, Equation C.23 represents a set of three equations, one for each component.

There are a couple of differential vector operators that occur frequently in chemical and physical problems. One of these is the *gradient*, which is defined by

$$\nabla f(x, y, z) = \text{grad}\, f(x, y, z) = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} \tag{C.24}$$

Note that the gradient operator, ∇, operates on a scalar function. The vector ∇f is called the *gradient vector* of $f(x, y, z)$. Consider a set of contour lines on a topographical map or a set of isotherms or isobars on a weather map or a set of equipotentials in a potential energy diagram. Those lines are collectively called *level curves*. If a surface is described by $z = f(x, y)$, then the level curves are given by $z =$ constant (Figure C.9). The path traced out by ∇f in Figure C.9 is normal (perpendicular) to each level curve that it crosses and follows the direction of steepest descent. For a set of equipotentials, for example, ∇f represents the corresponding electric field and traces out the path that a charged particle will follow (Figure C.10).

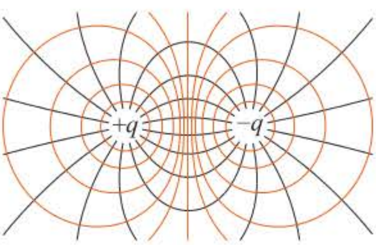

FIGURE C.10
The equipotentials (orange) and the electric field (black) of an electric dipole formed by equal and opposite charges.

Many physical laws are expressed in terms of a gradient vector. For example, Fick's law of diffusion says that the flux of a solute is proportional to the gradient of its concentration, or if $c(x, y, z, t)$ is the concentration of solute at the point (x, y, z) at time t, then

$$\text{flux of solute} = -D\nabla c(x, y, z, t)$$

where D is called the diffusion constant. Similarly, Fourier's law of heat flow says that the flux of heat is described by

$$\text{flux of heat} = -\lambda \nabla T(x, y, z, t)$$

where T is the temperature and λ is the thermal conductivity. If $V(x, y, z)$ is a mechanical potential energy experienced by a body, then the force on the body is given by

$$\mathbf{F} = -\nabla V(x, y, z) \tag{C.25}$$

In addition, if $\phi(x, y, z)$ is an electrostatic potential, then the electric field associated with that potential is given by

$$\mathbf{E} = -\nabla\, \phi(x, y, z) \tag{C.26}$$

EXAMPLE C–6
Suppose that a particle experiences a potential energy

$$V(x, y, z) = \frac{k_x x^2}{2} + \frac{k_y y^2}{2} + \frac{k_z z^2}{2},$$

where the k's are constant. Derive an expression for the force acting on the particle.

SOLUTION: We use Equation C.25 to write

$$\begin{aligned}\mathbf{F}(x, y, z) &= -\mathbf{i}\frac{\partial V}{\partial x} - \mathbf{j}\frac{\partial V}{\partial y} - \mathbf{k}\frac{\partial V}{\partial z} \\ &= -\mathbf{i}\, k_x x - \mathbf{j}\, k_y y - \mathbf{k}\, k_z z\end{aligned}$$

Problems

C–1. Find the length of the vector $\mathbf{v} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$.

C–2. Find the length of the vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ and of the vector $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

C–3. Prove that $\mathbf{u} \cdot \mathbf{v} = 0$ if $\mathbf{u}$ and $\mathbf{v}$ are perpendicular to each other. Two vectors that are perpendicular to each other are said to be orthogonal.

C–4. Show that the vectors $\mathbf{u} = 2\,\mathbf{i} - 4\,\mathbf{j} - 2\,\mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} + 4\,\mathbf{j} - 5\,\mathbf{k}$ are orthogonal.

C–5. Show that the vector $\mathbf{r} = 2\,\mathbf{i} - 3\,\mathbf{k}$ lies entirely in a plane perpendicular to the y axis.

C–6. Find the angle between the two vectors $\mathbf{u} = -\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$.

C–7. Show that the set of vectors $(1/\sqrt{3},\ 1/\sqrt{3},\ 0,\ 1/\sqrt{3})$, $(1/\sqrt{3},\ -1/\sqrt{3},\ 1/\sqrt{3},\ 0)$, $(0,\ 1/\sqrt{3},\ 1/\sqrt{3},\ -1/\sqrt{3})$, and $(1/\sqrt{3},\ 0,\ -1/\sqrt{3},\ -1/\sqrt{3})$ is orthonormal.

C–8. Determine $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ given that $\mathbf{u} = -\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}$. What is $\mathbf{v} \times \mathbf{u}$ equal to?

C–9. Show that $\mathbf{u} \times \mathbf{u} = 0$.

C–10. Using Equation C.16, prove that $\mathbf{u} \times \mathbf{v}$ is given by Equation C.17.

C–11. Show that $l = |\mathbf{l}| = mvr$ for circular motion.

C–12. Show that

$$\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v}) = \frac{d\mathbf{u}}{dt} \cdot \mathbf{v} + \mathbf{u} \cdot \frac{d\mathbf{v}}{dt}$$

and

$$\frac{d}{dt}(\mathbf{u} \times \mathbf{v}) = \frac{d\mathbf{u}}{dt} \times \mathbf{v} + \mathbf{u} \times \frac{d\mathbf{v}}{dt}$$

C–13. Using the results of Problem C–12, prove that

$$\mathbf{u} \times \frac{d^2\mathbf{u}}{dt^2} = \frac{d}{dt}\left(\mathbf{u} \times \frac{d\mathbf{u}}{dt}\right)$$

C–14. In vector notation, Newton's equations for a single particle are

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}(x,\ y,\ z)$$

By operating on this equation from the left by $\mathbf{r}\,\times$ and using the result of Problem C–13, show that

$$m\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \mathbf{r} \times \mathbf{F}$$

Because momentum is defined as $\mathbf{p} = m\mathbf{v} = m\dfrac{d\mathbf{r}}{dt}$, the above expression reads

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \mathbf{F}$$

But $\mathbf{r} \times \mathbf{p} = \mathbf{l}$, the angular momentum, and so we have

$$\frac{d\mathbf{l}}{dt} = \mathbf{r} \times \mathbf{F}$$

This is the form of Newton's equation for a rotating system. Notice that $d\mathbf{l}/dt = 0$, or that angular momentum is conserved if $\mathbf{r} \times \mathbf{F} = 0$. Can you identify $\mathbf{r} \times \mathbf{F}$?

C–15. Find the gradient of $f(x, y, z) = x^2 - yz + xz^2$ at the point (1, 1, 1).

C–16. The electrostatic potential produced by a dipole moment $\boldsymbol{\mu}$ located at the origin and directed along the x axis is given by

$$\phi(x, y, z) = \frac{\mu x}{(x^2 + y^2 + z^2)^{3/2}} \qquad (x, y, z \neq 0)$$

Derive an expression for the electric field associated with this potential.

C–17. We proved the *Schwartz inequality* for complex numbers in Problem A–16. For vectors, the Schwartz inequality takes the form

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq |\mathbf{u}|^2 |\mathbf{v}|^2$$

Why do you think that this is so? Do you see a parallel between this result for two-dimensional vectors and the complex number version?

C–18. We proved the *triangle inequality* for complex numbers in Problem A–17. For vectors, the triangle inequality takes the form

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|$$

Prove this inequality by starting with

$$|\mathbf{u} + \mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + 2\mathbf{u} \cdot \mathbf{v}$$

and then using the Schwartz inequality (previous problem). Why do you think this is called the triangle inequality?