MATHCHAPTER G

Matrices

Many physical operations such as magnification, rotation, and reflection through a plane can be represented mathematically by quantities called matrices. A matrix is a two-dimensional array that obeys a certain set of rules called *matrix algebra*. Even if matrices are entirely new to you, they are so convenient that learning some of their simpler properties is worthwhile. Furthermore, a great deal of quantum mechanics or modern quantum chemistry uses matrix algebra frequently.

Consider the lower of the two vectors shown in Figure G.1. The x and y components of the vector are given by $x_1 = r\cos\alpha$ and $y_1 = r\sin\alpha$, where r is the length of $\mathbf{r}_1$. Now let's rotate the vector counterclockwise through an angle θ, so that $x_2 = r\cos(\alpha+\theta)$ and $y_2 = r\sin(\alpha+\theta)$ (see Figure G.1). Using trigonometric formulas, we can write

$$x_2 = r\cos(\alpha+\theta) = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$$

$$y_2 = r\sin(\alpha+\theta) = r\cos\alpha\sin\theta + r\sin\alpha\cos\theta$$

or

$$\begin{aligned} x_2 &= x_1\cos\theta - y_1\sin\theta \\ y_2 &= x_1\sin\theta + y_1\cos\theta \end{aligned} \tag{G.1}$$

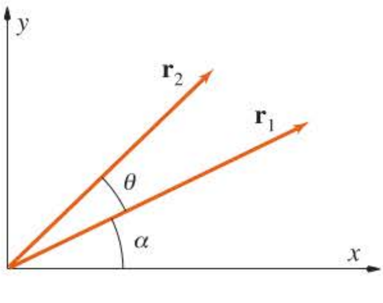

FIGURE G.1
An illustration of the rotation of a vector $\mathbf{r}$ through an angle θ.

We can display the set of coefficients here in the form

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{G.2}$$

We have expressed R in the form of a *matrix*, which is an array of numbers (or functions in this case) that obeys the rules of matrix algebra. We will denote a matrix by a sans serif symbol (e.g., A, B, etc.). Unlike determinants (MathChapter F), matrices do not have to be square arrays. Furthermore, unlike determinants, matrices cannot be reduced to a single number. The matrix R in Equation G.2 corresponds to a rotation through an angle θ.

The entries in a matrix A are called its *matrix elements* and are denoted by a_{ij}, where, as in the case of determinants, i designates the row and j designates the column. Two matrices, A and B, are equal if and only if they are of the same dimension and $a_{ij} = b_{ij}$ for all i and j. In other words, equal matrices are identical. Matrices can be added or subtracted only if they have the same number of rows and columns, in which case the elements of the resultant matrix are given by $a_{ij} + b_{ij}$. Thus, if

$$\mathsf{A} = \begin{pmatrix} -3 & 6 & 4 \\ 1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 2 & 1 & 1 \\ -6 & 4 & 3 \end{pmatrix}$$

then

$$\mathsf{C} = \mathsf{A} + \mathsf{B} = \begin{pmatrix} -1 & 7 & 5 \\ -5 & 4 & 5 \end{pmatrix}$$

If we write

$$\mathsf{A} + \mathsf{A} = 2\mathsf{A} = \begin{pmatrix} -6 & 12 & 8 \\ 2 & 0 & 4 \end{pmatrix}$$

we see that scalar multiplication of a matrix means that each element is multiplied by the scalar. Thus,

$$c\mathsf{M} = \begin{pmatrix} cM_{11} & cM_{12} \\ cM_{21} & cM_{22} \end{pmatrix} \tag{G.3}$$

EXAMPLE G–1

Using the matrices A and B above, form the matrix $\mathsf{D} = 3\mathsf{A} - 2\mathsf{B}$.

SOLUTION:

$$\begin{aligned} \mathsf{D} &= 3\begin{pmatrix} -3 & 6 & 4 \\ 1 & 0 & 2 \end{pmatrix} - 2\begin{pmatrix} 2 & 1 & 1 \\ -6 & 4 & 3 \end{pmatrix} \\ &= \begin{pmatrix} -9 & 18 & 12 \\ 3 & 0 & 6 \end{pmatrix} - \begin{pmatrix} 4 & 2 & 2 \\ -12 & 8 & 6 \end{pmatrix} = \begin{pmatrix} -13 & 16 & 10 \\ 15 & -8 & 0 \end{pmatrix} \end{aligned}$$

One of the most important aspects of matrices is matrix multiplication. For simplicity, we will discuss the multiplication of square matrices first. Consider some linear transformation of (x_1, y_1) into (x_2, y_2):

$$\begin{aligned} x_2 &= a_{11}x_1 + a_{12}y_1 \\ y_2 &= a_{21}x_1 + a_{22}y_1 \end{aligned} \tag{G.4}$$

represented by the matrix

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{G.5}$$

Now let's transform (x_2, y_2) into (x_3, y_3):

$$\begin{aligned} x_3 &= b_{11}x_2 + b_{12}y_2 \\ y_3 &= b_{21}x_2 + b_{22}y_2 \end{aligned} \tag{G.6}$$

represented by the matrix

$$\mathsf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{G.7}$$

Let the transformation of (x_1, y_1) directly into (x_3, y_3) be given by

$$\begin{aligned} x_3 &= c_{11}x_1 + c_{12}y_1 \\ y_3 &= c_{21}x_1 + c_{22}y_1 \end{aligned} \tag{G.8}$$

represented by the matrix

$$\mathsf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \tag{G.9}$$

Symbolically, we can write that

$$\mathsf{C} = \mathsf{BA}$$

because C results from transforming from (x_1, y_1) to (x_2, y_2) by means of A, followed by transforming (x_2, y_2) to (x_3, y_3) by means of B. Let's find the relation between the elements of C and those of A and B. Substitute Equations G.4 into G.6 to obtain

$$\begin{aligned} x_3 &= b_{11}(a_{11}x_1 + a_{12}y_1) + b_{12}(a_{21}x_1 + a_{22}y_1) \\ y_3 &= b_{21}(a_{11}x_1 + a_{12}y_1) + b_{22}(a_{21}x_1 + a_{22}y_1) \end{aligned} \tag{G.10}$$

or

$$\begin{aligned} x_3 &= (b_{11}a_{11} + b_{12}a_{21})x_1 + (b_{11}a_{12} + b_{12}a_{22})y_1 \\ y_3 &= (b_{21}a_{11} + b_{22}a_{21})x_1 + (b_{21}a_{12} + b_{22}a_{22})y_1 \end{aligned}$$

Thus, we see that

$$\mathsf{C} = \mathsf{BA} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix} \quad \text{(G.11)}$$

This result may look complicated, but it has a nice pattern that we will illustrate two ways. Mathematically, the ijth element of C is given by the formula

$$c_{ij} = \sum_k b_{ik}a_{kj} \quad \text{(G.12)}$$

Notice that we sum over the middle index. For example,

$$c_{11} = \sum_k b_{1k}a_{k1} = b_{11}a_{11} + b_{12}a_{21}$$

as in Equation G.11. A more pictorial way is to notice that any element in C can be obtained by multiplying elements in any row in B by the corresponding elements in any column in A, adding them, and then placing them in C where the row and column intersect. In terms of vectors, we take the dot product of the row of B and the column of A and place the result at their intersection. For example, c_{11} is obtained by multiplying the elements of row 1 of B with the elements of column 1 of A, or by the scheme

$$\begin{array}{c} \rightarrow \\ \end{array} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \overset{\downarrow}{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}} = \begin{pmatrix} b_{11}a_{11} + b_{12}a_{21} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

and c_{12} by

$$\begin{array}{c} \rightarrow \\ \end{array} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} a_{11} & \overset{\downarrow}{a_{12}} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cdot & b_{11}a_{12} + b_{12}a_{22} \\ \cdot & \cdot \end{pmatrix}$$

EXAMPLE G–2

Find $\mathsf{C} = \mathsf{BA}$ if

$$\mathsf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \quad \text{and} \quad \mathsf{A} = \begin{pmatrix} -3 & 0 & -1 \\ 1 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

SOLUTION:

$$\mathsf{C} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 & -1 \\ 1 & 4 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -3+2+1 & 0+8+1 & -1+0+1 \\ -9+0-1 & 0+0-1 & -3+0-1 \\ 3-1+2 & 0-4+2 & 1+0+2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 9 & 0 \\ -10 & -1 & -4 \\ 4 & -2 & 3 \end{pmatrix}$$

EXAMPLE G–3

The matrix R given by Equation G.2 represents a rotation through the angle θ. Show that R^2 represents a rotation through an angle 2θ. In other words, show that R^2 represents two sequential applications of R.

SOLUTION:

$$\mathsf{R}^2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{pmatrix}$$

Using standard trigonometric identities, we get

$$\mathsf{R}^2 = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}$$

which represents rotation through an angle 2θ.

Matrices do not have to be square to be multiplied together, but either Equation G.11 or the pictorial method illustrated above suggests that the number of columns of B must be equal to the number of rows of A. When this is so, A and B are said to be *compatible*. For example, Equations G.4 can be written in matrix form as

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{G.13}$$

An important aspect of matrix multiplication is that BA does not necessarily equal AB. For example, if

$$\mathsf{A} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathsf{B} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$\mathsf{AB} = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 3 & 0 \end{pmatrix}$$

and

$$\mathsf{BA} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 6 \\ -1 & 0 \end{pmatrix}$$

and so $\mathsf{AB} \neq \mathsf{BA}$. If it does happen that $\mathsf{AB} = \mathsf{BA}$, then A and B are said to *commute*.

EXAMPLE G–4

Do the matrices A and B commute if

$$\mathsf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathsf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

SOLUTION:

$$\mathsf{AB} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$$

and

$$\mathsf{BA} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

so they do not commute.

Another property of matrix multiplication that differs from ordinary scalar multiplication is that the equation

$$\mathsf{AB} = \mathsf{O}$$

where O is the zero matrix (all elements equal to zero) does not imply that A or B necessarily is a zero matrix. For example,

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A linear transformation that leaves (x_1, y_1) unaltered is called the identity transformation, and the corresponding matrix is called the *identity matrix* or the *unit matrix*. All the elements of the unit matrix are equal to zero, except those along the diagonal, which equal one:

$$\mathsf{I} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

The elements of I are δ_{ij}, the Kronecker delta, which equals one when $i = j$ and zero when $i \neq j$. The unit matrix has the property that

$$\mathsf{IA} = \mathsf{AI} \tag{G.14}$$

The unit matrix is an example of a *diagonal matrix*. The only nonzero elements of a diagonal matrix are along its diagonal. Diagonal matrices are necessarily square matrices.

If $\mathsf{BA} = \mathsf{AB} = \mathsf{I}$, then B is said to be the *inverse* of A, and is denoted by A^{-1}. Thus, A^{-1} has the property that

$$\mathsf{AA}^{-1} = \mathsf{A}^{-1}\mathsf{A} = \mathsf{I} \tag{G.15}$$

If A represents some transformation, then A^{-1} undoes that transformation and restores the original state. There are recipes for finding the inverse of a matrix, but we won't need them (see Problem G–9, however). Nevertheless, it should be clear on physical grounds that the inverse of R in Equation G.2 is

$$\mathsf{R}^{-1} = \mathsf{R}(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{G.16}$$

which is obtained from R by replacing θ by $-\theta$. In other words, if $\mathsf{R}(\theta)$ represents a rotation through an angle θ, then $\mathsf{R}^{-1} = \mathsf{R}(-\theta)$ and represents the reverse rotation. It is easy to show that R and R^{-1} satsify Equation G.15. Using Equations G.2 and G.16, we have

$$\begin{aligned} \mathsf{R}^{-1}\mathsf{R} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned}
\mathsf{RR}^{-1} &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

Most of the matrices that occur in quantum mechanics are *orthogonal*. The characteristic property of an orthogonal matrix is that both its columns and its rows form a set of orthogonal vectors. For example,

$$\mathsf{A} = \frac{1}{9}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix} \tag{G.17}$$

is an orthogonal matrix. The vectors that make up the rows, for example, are normalized,

$$\frac{1}{81}[1^2 + 8^2 + (-4)^2] = 1$$

$$\frac{1}{81}[4^2 + (-4)^2 + (-7)^2] = 1$$

and

$$\frac{1}{81}(8^2 + 1^2 + 4^2) = 1$$

and the dot products of its rows are

$$\begin{aligned}
1 \times 4 + 8 \times (-4) + (-4)(-7) &= 0 \\
1 \times 8 + 8 \times 1 + (-4)(4) &= 0 \\
4 \times 8 + (-4)(1) + (-7)(4) &= 0
\end{aligned}$$

The same is true for its columns.

It is very easy to find the inverse of an orthogonal matrix. First, we define the transpose of A, A^T, as the matrix that we obtain from A by interchanging rows and columns. The transpose of Equation G.17 is

$$\mathsf{A}^\mathsf{T} = \begin{pmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{pmatrix} \tag{G.18}$$

We obtain A^{T} from A by simply flipping it about its diagonal. In terms of the elements of A, we have $a_{ij}^{\mathsf{T}} = a_{ji}$. If $\mathsf{A}^{\mathsf{T}} = \mathsf{A}$, then the matrix is said to be *symmetric*. In terms of the elements of A, we have $a_{ij} = a_{ji}$. Most matrices in quantum mechanics are symmetric. It turns out that the inverse of an orthogonal matrix is equal to its transpose, or

$$\mathsf{A}^{-1} = \mathsf{A}^{\mathsf{T}} \qquad \text{(orthogonal matrix)} \tag{G.19}$$

EXAMPLE G–5

Show that A^{T} given by Equation G.18 is equal to A^{-1}.

SOLUTION:

$$\mathsf{A}^{\mathsf{T}}\mathsf{A} = \frac{1}{81}\begin{pmatrix} 1 & 4 & 8 \\ 8 & -4 & 1 \\ -4 & -7 & 4 \end{pmatrix}\begin{pmatrix} 1 & 8 & -4 \\ 4 & -4 & -7 \\ 8 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice that the products of the rows of A^{T} into the columns of A are exactly the same as those of our original definition of an orthogonal matrix. Thus, $\mathsf{A}^{\mathsf{T}} = \mathsf{A}^{-1}$ because both the rows and the columns of A form sets of orthogonal vectors. Orthogonal matrices correspond to rotations in space. When an orthogonal matrix A operates on a vector $\mathbf{v}$, it simply rotates the vector.

We can associate a determinant with a square matrix by writing

$$\det \mathsf{A} = |\mathsf{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Thus, the determinant of R is

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} = \cos^2\theta + \sin^2\theta = 1$$

and $\det \mathsf{R}^{-1} = 1$ also. If $\det \mathsf{A} = 0$, then A is said to be a *singular matrix*. Singular matrices do not have inverses.

A quantity that frequently arises in group theory is the sum of the diagonal elements of a matrix, called the *trace* of the matrix. Thus, the trace of the matrix

$$\mathsf{B} = \begin{pmatrix} 1/2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1/2 \end{pmatrix}$$

is 3, which we write as $\mathrm{Tr}\,\mathsf{B} = 3$ (Problem G–15).

Problems

G–1. Given the two matrices

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} -1 & 1 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}$$

form the matrices $\mathsf{C} = 2\,\mathsf{A} - 3\,\mathsf{B}$ and $\mathsf{D} = 6\,\mathsf{B} - \mathsf{A}$.

G–2. Given the three matrices

$$\mathsf{A} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \mathsf{B} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \mathsf{C} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

show that $\mathsf{A}^2 + \mathsf{B}^2 + \mathsf{C}^2 = \frac{3}{4}\mathsf{I}$, where I is a unit matrix. Also show that

$$\mathsf{AB} - \mathsf{BA} = i\mathsf{C}$$

$$\mathsf{BC} - \mathsf{CB} = i\mathsf{A}$$

$$\mathsf{CA} - \mathsf{AC} = i\mathsf{B}$$

G–3. Given the matrices

$$\mathsf{A} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathsf{B} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \mathsf{C} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

show that

$$\mathsf{AB} - \mathsf{BA} = i\mathsf{C}$$

$$\mathsf{BC} - \mathsf{CB} = i\mathsf{A}$$

$$\mathsf{CA} - \mathsf{AC} = i\mathsf{B}$$

and

$$\mathsf{A}^2 + \mathsf{B}^2 + \mathsf{C}^2 = 2\mathsf{I}$$

where I is a unit matrix.

G–4. Do you see any similarity between the results of Problems G–2 and G–3 and the commutation relations involving the components of angular momentum?

G–5. A three-dimensional rotation about the z axis can be represented by the matrix

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Show that

$$\det \mathsf{R} = |\mathsf{R}| = 1$$

Also show that

$$\mathsf{R}^{-1} = \mathsf{R}(-\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

G–6. Show that the matrix R in Problem G–5 is orthogonal.

G–7. Given the matrices

$$\mathsf{C}_3 = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad \sigma_{\mathsf{v}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma'_{\mathsf{v}} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \sigma''_{\mathsf{v}} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

show that

$$\sigma_{\mathsf{v}}\mathsf{C}_3 = \sigma''_{\mathsf{v}} \qquad \mathsf{C}_3\sigma_{\mathsf{v}} = \sigma'_{\mathsf{v}}$$

$$\sigma''_{\mathsf{v}}\sigma'_{\mathsf{v}} = \mathsf{C}_3 \qquad \mathsf{C}_3\sigma''_{\mathsf{v}} = \sigma_{\mathsf{v}}$$

Calculate the determinant associated with each matrix. Calculate the trace of each matrix.

G–8. Which of the matrices in Problem G–7 are orthogonal?

G–9. The inverse of a matrix A can be found by using the following procedure:

1. Replace each element of A by its cofactor in the corresponding determinant (see MathChapter F for a definition of a cofactor).
2. Take the transpose of the matrix obtained in step 1.
3. Divide each element of the matrix obtained in step 2 by the determinant of A.

For example, if

$$\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

then det $\mathsf{A} = -2$ and

$$\mathsf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

Show that $\mathsf{A}\mathsf{A}^{-1} = \mathsf{A}^{-1}\mathsf{A} = \mathsf{I}$. Use the above procedure to find the inverse of

$$\mathsf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \qquad \text{and} \qquad \mathsf{B} = \begin{pmatrix} 0 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

G–10. Recall that a singular matrix is one whose determinant is equal to zero. Referring to the procedure in Problem G–9, do you see why a singular matrix has no inverse?

G–11. Consider the matrices A and S,

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad \mathsf{S} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

First, show that S is orthogonal. Then evaluate the matrix $\mathsf{D} = \mathsf{S}^{-1}\mathsf{A}\mathsf{S} = \mathsf{S}^{\mathsf{T}}\mathsf{A}\mathsf{S}$. What form does D have?

G–12. A matrix whose elements satisfy the relation $a_{ij} = a^*_{ji}$ is called *Hermitian*. You can think of a Hermitian matrix as a symmetric matrix in a complex space. Show that the eigenvalues of a Hermitian matrix are real. (Note the similarity between a Hermitian operator and a Hermitian matrix.) *Hint*: Start with $\mathsf{H}\mathbf{x}_i = \lambda_i \mathbf{x}_i$ and $\mathsf{H}^*\mathbf{x}^*_j = \lambda^*_j \mathbf{x}^*_j$ and multiply the first equation from the left by $\mathbf{x}^*_j$ and the second from the left by $\mathbf{x}_i$ and then use the Hermitian property of H.

G–13. Show that $(\mathsf{AB})^{\mathsf{T}} = \mathsf{B}^{\mathsf{T}}\mathsf{A}^{\mathsf{T}}$.

G–14. Show that $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1}$.

G–15. Show that $\mathrm{Tr}\,\mathsf{AB} = \mathrm{Tr}\,\mathsf{BA}$.

G–16. Consider the simultaneous algebraic equations

$$x + y = 3$$
$$4x - 3y = 5$$

Show that this pair of equations can be written in the matrix form

$$\mathsf{A}\mathbf{x} = \mathbf{c} \tag{1}$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{c} = \begin{pmatrix} 3 \\ 5 \end{pmatrix} \qquad \text{and} \qquad \mathsf{A} = \begin{pmatrix} 1 & 1 \\ 4 & -3 \end{pmatrix}$$

Now multiply equation 1 from the left by A^{-1} to obtain

$$\mathbf{x} = \mathsf{A}^{-1}\mathbf{c} \tag{2}$$

Now show that

$$\mathsf{A}^{-1} = -\frac{1}{7}\begin{pmatrix} -3 & -1 \\ -4 & 1 \end{pmatrix}$$

and that

$$\mathbf{x} = -\frac{1}{7}\begin{pmatrix} -3 & -1 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

or that $x = 2$ and $y = 1$. Do you see how this procedure generalizes to any number of simultaneous equations?

G–17. Solve the following simultaneous algebraic equations by the matrix inverse method developed in Problem G–16:

$$\begin{aligned} x + y - z &= 1 \\ 2x - 2y + z &= 6 \\ x + 3z &= 0 \end{aligned}$$

First, show that

$$\mathsf{A}^{-1} = \frac{1}{13}\begin{pmatrix} 6 & 3 & 1 \\ 5 & -4 & 3 \\ -2 & -1 & 4 \end{pmatrix}$$

and evaluate $\mathbf{x} = \mathsf{A}^{-1}\mathbf{c}$.